quant-ph/9611012

Modern Physics Letters A, Vol. 11, No. 19 (1996) 1563-1567

NEW FEATURES IN SUPERSYMMETRY BREAKDOWN IN QUANTUM MECHANICS

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Abstract

The supersymmetric quantum mechanical model based on higher-derivative supercharge operators possessing unbroken supersymmetry and discrete energies below the vacuum state energy is described. As an example harmonic oscillator potential is considered

1. Ideas of the supersymmetry have appeared in physics for the first time in the quantum field theory for unifying the interactions of a different nature [1]. In a supersymmetric theory the supersymmetry can be either exact or spontaneously broken. The supersymmetric quantum mechanics has been introduced [2] to illustrate the problems of the supersymmetry breakdown in supersymmetric quantum field theories. For this purpose the Witten criterion based on the Witten index [2] has been elaborated. In the case of the broken supersymmetry the entire spectrum of the super-Hamiltonian is twofold degenerate and in the case of the exact one its vacuum state is nondegenerate. In the first case the supercharge operators map the two states corresponding to the vacuum energy (zero energy) one into another and in the second one the vacuum state is annihilated by both supercharges. (See for example a recent survey [3].)

Recently higher-order derivative extension of the supersymmetric quantum mechanics has been elaborated [4]. In this approach supercharges are constructed in terms of the higher-derivative differential operators and the corresponding superalgebra is polynomial in the Hamiltonian. This model exhibits a number of unusual

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properties [5]. In particular, the Witten criterion of spontaneous supersymmetry breaking is no longer applicable [4].

We now want to describe an unusual property of such models in terms of supersymmetry breakdown which has not been described earlier. In our case the state which is nondegenerate and annihilated by both mutually conjugated supercharges is situated in the middle of the discrete spectrum of a super-Hamiltonian. It follows that if one associates with this state the zero energy value, the underlying energies should take negative values. This situation will not occur in the conventional supersymmetric quantum mechanics [3] and we can claim that our higher-derivative model exhibits at once the properties of the models with both exact and spontaneously broken supersymmetry.

2.The higher-derivative supersymmetry in quantum mechanics [4] is closely related to the higher-derivative Darboux transformation [5], [6], [7]. This transformation, denoted here as $L^{(N)}$, is introduced in accordance with the general conception of the transformation operators [8] as an N-order differential operator intertwining two Hamiltonians h_0 and h_N. The proper functions $\varphi_E(x)$ of one of them (for example h_0) are assumed to be known: $h_0\varphi_E(x) = E\varphi_E(x)$. One then obtains the proper function $\psi_E(x)$, corresponding to the same eigenvalue E, of the other (i.e. h_N) with the help of the operator $L^{(N)}$: $\psi_E = L^{(N)}\varphi_E$, $h_N\psi_E(x) = E\psi_E(x)$ except for the functions which form the kernel of the operator Laplace adjoint to $L^{(N)}$ denoted by $L^{(N)^+}$. We assume that the operators h_0 and h_N are self-adjoint. (More precisely we suppose that their potentials are real-valued functions and the Hamiltonians are essentially self-adjoint in the sense of some scalar product). In this case operator $L^{(N)^+}$ assures the transformation in the inverse direction: from the eigenfunctions ψ_E to the eigenfunctions φ_E. When $N = 1$ we have the well-known Darboux transformation [9] called *first-order Dardoux transformation.*

It can be shown [6] that the operator $L^{(N)}$ can always be presented as a product of N first-order Darboux transformation operators between every two juxtaposed Hamiltonians $h_0, h_1, \ldots, h_N$: $L^{(N)} = L_{0,1}L_{1,2}\ldots L_{N-1,N}$, $L_{p.p+1}h_p = h_{p+1}L_{p,p+1}$, $p = 0, 1, \ldots, N-1$. Some of the intermediate Hamiltonians h_p can have complex-valued potentials but the final potential of the h_N remains always real-valued function (so-called irreducible case [4])

In this latter we want to point out that other than that described in Ref. [4] irreducible case exists. It is connected with the choice of discrete spectrum functions of the Hamiltonian h_0 as the transformation functions. In this case the intermediate potentials are real-valued functions having additional singularities with respect to initial potential.

It follows from theorem proved in Ref. [6] that the operator $L^{(N)}$ can always be

presented in the form known as Crum-Krein formula [10], [11]:

$$L^{(N)} = W^{-1}(u_1, u_2, \ldots, u_N) \begin{vmatrix} u_1 & u_2 & \cdots & 1 \\ u_1' & u_2' & \cdots & d/dx \\ \vdots & \vdots & \ddots & \vdots \\ u_1^{(N)} & u_2^{(N)} & \cdots & d^N/dx^N \end{vmatrix} \tag{1}$$

where W stands for the usual symbol for the Wronskian of the functions $u_1, u_2, \ldots, u_N$ called transformation functions and satisfied the initial Schrödinger equation ($h_0 u_i = \alpha_i u_i$), the prime denotes the derivative with respect to real coordinate x, and the determinant is a differential operator obtained by the development of the determinant in the last column with the functional coefficients placed before the derivative operators. Potential difference between the final Schrödinger equation potential and the initial one reads as follows: $A_N(x) = V_N(x) - V_0(x) = -2[\log W(u_1, u_2, \ldots, u_N)]''$.

The function $A_N(x)$ is well defined if the Wronskian $W(u_1, u_2, \ldots, u_N)$ conserves its sign in the interval $R = [a, b]$ for the variable x in the initial Schrödinger equation. If the discrete spectrum eigenfunctions u_i of the Hamiltonian h_0 are enumerated by the number of their zeros, the condition for the Wronskian to conserve its sign is formulated by Krein [11]: the Wronskian $W(u_{k_1}, u_{k_2}, \ldots, u_{k_N})$ conserves its sign, the integers k_i being equal to the number of zeros of functions u_{k_i}, if for all $k = 0, 1, 2, \ldots$, the following inequality: $(k - k_1)(k - k_2)\ldots(k - k_N) \geq 0$ holds. In particular, the functions u_{k_i} may be two-by-two juxtaposed discrete spectrum eigenfunctions. The levels with $E = E_{k_i}$, $i = 1, \ldots, N$ will be absent in the discrete spectrum of the new Hamiltonian h_N.

It follows from the formula (1) that $\ker L^{(N)} = \text{span}\{u_i\}$. For $\ker L^{(N)^+}$ we have: $\ker L^{(N)^+} = \text{span}\{v_i\}$ where [6] $v_k = W^{(k)}(u_1, u_2, \ldots, u_N)W^{-1}(u_1, u_2, \ldots, u_N)$, $W^{(k)}(u_1, u_2, \ldots, u_N)$ is the $N-1$-order Wronskian constructed from the functions $u_1, u_2, \ldots, u_N$ except for the u_k, $k = 1, \ldots, N$.

The product $L^{(N)^+}L^{(N)}$ being a symmetry operator for the initial Schrödinger equation is a polynomial function of the initial Hamiltonian. Taking into account the condition $L^{(N)}u_i = 0$, $i = 1, \ldots, N$ we obtain more precisely [4], [6]:

$$L^{(N)^+}L^{(N)} = \prod_{i=1}^{N}(h_0 - \alpha_i).$$

The same is true for the product $L^{(N)}L^{(N)^+}$:

$$L^{(N)}L^{(N)^+} = \prod_{i=1}^{N}(h_N - \alpha_i).$$

3. Let u_i be two-by-two juxtaposed discrete spectrum eigenfunctions of h_0 and $\{\varphi_i, \varphi_\lambda\}$ be a basis set of the Hilbert space $H(R)$ ($\{\varphi_i\}$ is a discrete subsystem and $\{\varphi_\lambda\}$ is a continuous one). Introduce the notation $N_0 = \{i : u_i = \varphi_i\}$. Then

the system of functions $\{L^{(N)}\varphi_i,\ L^{(N)}\varphi_\lambda,\ i \notin N_0\}$ is complete in $H(R)$ [11], [12]. With the help of $L^{(N)}$ and $L^{(N)^+}$ we built up the supercharges $Q = \begin{pmatrix} 0 & 0 \\ L^{(N)} & 0 \end{pmatrix} = (Q^+)^\dagger$ which together with the super-Hamiltonian $\mathcal{H} = \mathrm{diag}(h_0, h_N)$ form an N-order superalgebra [4], [6]:

$$[Q, \mathcal{H}] = [Q^+, \mathcal{H}] = 0, \{Q, Q^+\} = \prod_{i=1}^{N} (\mathcal{H} - \alpha_i).$$

Every energy $E = E_i$ of the superhamiltonian $\mathcal{H}$ is twofold degenerate if $i \notin N_0$ and nondegenerate if $i \in N_0$. The energy $E_{i_0} = \min\limits_{i \in N_0}\{E_i\}$ can be associated with the ground state of the super-Hamiltonian $\mathcal{H}$ and the wave function $\Psi_{i_0} = (\varphi_{i_0}, 0)^T$ being annihilated by both supercharges can be considered as the vacuum state. All the other nondegenerate states $\Psi_i = (\varphi_i, 0)^T$, $i \in N_0$ are also annihilated by both supercharges. We can choose the set $\{u_i, i \in N_0\}$ in such a way that the ground state of the Hamiltonian h_0 does not belong to this set. In this case proper functions of the super-Hamiltonian with the energies below to its vacuum state exist and are twofold degenerate.

4. We will cite an example of the above-described situation. Consider the harmonic potential $h_0 = -d^2/dx^2 + x^2/4 - 1/2$ with the discrete spectrum $E_n = n = 0, 1, 2, \ldots$ and the well-known discrete spectrum eigenfunctions $\varphi_n = (\sqrt{2\pi} n!)^{-1/2} \times \exp(-x^2/4) He_n(x)$ where $He_n(x)$ is the Hermit polynomial [13]. The double Darboux transformation with the juxtaposed functions φ_k and φ_{k+1}, $k \geq 0$ produces a new potential of the form [6]:

$$V_2(x) = \tfrac{x^2}{4} + \tfrac{3}{2} - 2\tfrac{J_k''(x)}{J_k(x)} + 2\left(\tfrac{J_k'(x)}{J_k(x)}\right)^2,$$
$$J_k(x) = \sum_{i=0}^{k} \tfrac{\Gamma(k+1)}{\Gamma(i+1)} He_i^2(x).$$

In its discrete spectrum the levels $E = k$ and $E = k+1$ are absent. The normalized to unity wave functions have the form

$$\psi_n(x) = [\sqrt{2\pi} n!(n-k)(n-k-1)]^{-1/2} \exp(-x^2/4)$$
$$\times [(n-k) He_n(x) + f_{kn}(x) \tfrac{He_{k+1}(x)}{J_k(x)}],$$
$$f_{kn}(x) = He_k(x) He_{n+1}(x) - He_n(x) He_{k+1}(x), \quad n \neq k, (k+1).$$

The state $\Psi_0 = (\varphi_k, 0)^T$ having the minimal energy value among the two states annihilated by both supercharges can be associated with the vacuum state. All the states $\Psi_{i-k} = a(\varphi_i, 0)^T + b(0, \psi_i)^T$, $i < k$, $a, b \in \mathbb{C}^1$ have the energies below energy $E = k$ of the vacuum state Ψ_0.

5. The supersymmetric quantum mechanics is now widely used in different branches of physics such as statistical physics, condensed matter, atomic physics [3], [14]. An essential ingredient of this theory is the Darboux transformation which

permits us to construct for every exactly solvable potential a family of its exactly solvable partners. If we start (as in our example) from the harmonic potential we can construct exactly solvable potentials with equidistant or quaziequidistant spectra [6]. Coherent states of these potentials [15] known in quantum optics as wavelets being constructed represent nondispersive wave packets.

The above considered potential in the particular case $k = 1$ has been obtained earlier by other means [16]. This potential corresponds to the one of the rational solutions of the Painlevé IV differential equation [17]. The connection of the Painlevé IV and V transcendents with the Schrödinger equation was studied in Ref. [18]. In our opinion with the help of the Darboux transformation it is possible to establish the correspondence between the known rational solutions of these equations [19] and exactly solvable potentials with the quaziequidistant spectra.

An application of the double Darboux transformation to the Coulomb potential which gives a new exactly solvable potential, in which the discrete spectrum the levels with $n = 2$ and $n = 3$ are absent, was made in Ref. [20]. Using these results and the above described approach we can construct for this system the second-order supersymmetric model analogous to the one discussed here.

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